

*Rapid Note***Anomalous scaling in the Zhang model**R. Pastor-Satorras<sup>1,a</sup> and A. Vespignani<sup>2</sup><sup>1</sup> Dept. de Física Fonamental, Facultat de Física, Universitat de Barcelona, Av. Diagonal 647, 08028 Barcelona, Spain<sup>2</sup> The Abdus Salam International Centre for Theoretical Physics (ICTP), PO Box 586, 34100 Trieste, Italy

Received 8 August 2000 and Received in final form 4 October 2000

**Abstract.** We apply the moment analysis technique to analyze large scale simulations of the Zhang sandpile model. We find that this model shows different scaling behavior depending on the update mechanism used. With the standard parallel updating, the Zhang model violates the finite-size scaling hypothesis, and it also appears to be incompatible with the more general multifractal scaling form. This makes impossible its affiliation to any one of the known universality classes of sandpile models. With sequential updating, it shows scaling for the size and area distribution. The introduction of stochasticity into the toppling rules of the parallel Zhang model leads to a scaling behavior compatible with the Manna universality class.

**PACS.** 05.70.Ln Nonequilibrium and irreversible thermodynamics – 05.65.+b Self-organized systems

The identification of universality classes is one of the most important and still open problems in the field of self-organized criticality (SOC) [1]. In spite of the great relevance of this issue, however, it has not been possible until very recently to clearly discern differences in the critical behavior of the various SOC models proposed so far. The situation seems now to have been settled in the case of the Bak-Tang-Wiesenfeld (BTW) [2] and the Abelian Manna [3,4] models that represent, respectively, the prototypical examples of deterministic and stochastic sandpile automata. In this particular case, recent large scale simulations [5–8] clearly indicate that Manna and BTW models belong to different universality classes.

The universality class asset remains still uncertain for the Zhang model [9], which is the archetype of all sandpile automata with continuous variables. This model has deterministic dynamics like the BTW model, and in contrast with most other cases, it is *non-Abelian*. This means that the stable configurations obtained at the end of an avalanche depend on the order in which the active sites are updated. In spite of this essential peculiarity, earlier simulations of the model [9] placed it in the same universality class as the BTW model, of which it was supposed to be the continuous counterpart. This conclusion was confirmed by the large scale simulations performed by Lübeck in  $d = 2$  [10]. On the other hand, Milshtein *et al.* [5], analyzing different magnitudes than Lübeck, arrived at the opposite result in  $d = 2$ , namely, they observed noticeable differences in the critical exponents. Finally, the simulations of Giacometti and Díaz-Guilera [11] provided

evidence that, even though the exponents of both models are similar in  $d = 2$ , they do not coincide in  $d = 3$ .

Recently, it has been shown [12,13] that the deterministic nature of the BTW model renders its scaling incompatible with the standard finite-size scaling (FSS) hypothesis, and induces moreover peculiar non-ergodic effects [8]. Thus, it would not be surprising to find similar anomalies in the Zhang model because of its deterministic nature. In this paper we characterize the Zhang universality class by applying the recently proposed moment analysis technique [12,13]. In the following we will show that the scaling of the Zhang model depends very strongly on the updating mechanism implemented in the simulations, either parallel or sequential. The Zhang model with parallel updating, as it has been customarily defined in the literature, displays a complex scaling behavior that is not compatible neither with the standard finite-size scaling (FSS) hypothesis nor with the multifractal picture [14] proposed for the avalanche distribution of the BTW model [13]. On the other hand, the Zhang model with sequential updating shows well defined size and area exponents. The origin of the complex behavior of the parallel Zhang model can be ascribed to the *deterministic* microscopic dynamical rules of the model. In order to prove this conjecture, we study a variation of the Zhang model, the stochastic parallel Zhang model, which exhibits a standard FSS behavior, compatible with the universality class of the Manna model.

We consider the definition of the original Zhang model [9]. On each site of a  $d$  dimensional hypercubic lattice of size  $L$  we assign a *continuous* variable  $E_i$ , called “energy”.

<sup>a</sup> romu@complex.upc.es

At each time step, an amount of energy  $\delta$  is added to a randomly selected site  $j$  ( $E_j \rightarrow E_j + \delta$ ). The quantity  $\delta$  is a random variable uniformly distributed in  $[0, \delta_{\max}]$  [15]. In our simulations we consider the fixed value  $\delta_{\max} = 0.25$ . When a site acquires an energy larger than or equal to 1, ( $E_k \geq 1$ ), it becomes active and topples. An active site  $k$  relaxes losing *all* its energy, which is equally distributed among its nearest neighbors:  $E_k \rightarrow 0$ , and  $E_{k'} \rightarrow E_{k'} + E_k/2d$ . Here the index  $k'$  runs over the set of all nearest neighbors of the site  $k$ . The transported energy can activate the nearest neighbor sites and thus create an avalanche. Energy can be lost only at the boundary of the system (open boundary conditions). The avalanche stops when all sites in the lattice are subcritical ( $E_i < 1$ ). Given these dynamical rules, it is easy to see that the Abelian nature of the model depends on the type of updating implemented. With *parallel* updating – parallel Zhang (P-Z) model – at each time step  $t$  in the evolution, all active sites are toppled simultaneously, and time is incremented  $t \rightarrow t + 1$ . Since all the energy of an active site is transferred to its nearest neighbors, we notice that in a bipartite lattice (such as the hypercubic lattice here considered) all active sites at a give time step  $t$  reside onto the same sublattice, and that activity alternates between the dual sublattices in consecutive time steps. Again, since all the energy is transferred in a toppling, the state of the active sites in a sublattice at time  $t$  is independent of the order in which the active sites in the dual sublattice were updated at time  $t - 1$ . On the other hand, with sequential updating – sequential Zhang (S-Z) model – at each time step  $t$  an active site is randomly chosen among all the  $N_a(t)$  active sites present at that time. The chosen site is toppled and time is incremented  $t \rightarrow t + 1/N_a(t)$ . In this case, activity is not restricted to alternate sublattices, but spreads all over the system. Depending on the order in which the intermediate list of over-critical sites is updated, any active site with at least one active nearest neighbor can end up with an energy  $E_i = 0$  (if its nearest neighbors are updated before it) or with energy  $E_i > 0$  (if it is updated before its nearest neighbors). In this case the model fully exploits its non-Abelian character.

We have analyzed both parallel and sequential Zhang models by determining their critical exponents [1]. In the limit of an infinite slow driving, *i.e.* the energy addition is interrupted during the avalanche evolution, defining a complete time scale separation [16], the system reaches a critical stationary state with avalanches of activity distributed according to power laws. If we define the probability distributions  $P(x)$  of occurrence of an avalanche of a given size  $s$ , time  $t$ , and area  $a$ , the FSS hypothesis [17], usually assumed in SOC systems, states that

$$P(x, L) = x^{-\tau_x} \mathcal{F}\left(\frac{x}{L^{\beta_x}}\right), \quad (1)$$

with  $x = s, t$ , or  $a$ , respectively. If the FSS ansatz is valid, then the critical exponents  $\beta_x$  and  $\tau_x$  completely determine the universality class of the model under scrutiny [18].

Previous numerical works on the Zhang model [9–11] have most often proceeded measuring the exponents as the

slope in a log-log plot of the density  $P(x, L)$  as a function of the magnitude  $x$ . Even though with this procedure one can determine the exponents within a 10% accuracy, its performance is affected by the existence of the upper and lower cutoffs, which render difficult its application. In this respect, it is better to use analysis techniques that contain explicitly the system-size dependency.

The moment analysis technique was introduced by De Menech *et al.* in the context of the two dimensional BTW model [12], and its validity has been extensively checked for both Abelian and stochastic models [6, 7]. In this method, the  $q$ th moment of a probability distribution on a lattice of size  $L$  is defined by  $\langle x^q \rangle_L = \int x^q P(x, L) dx$ . Assuming the FSS hypothesis, equation (1), the  $q$ th moment has the size dependence:

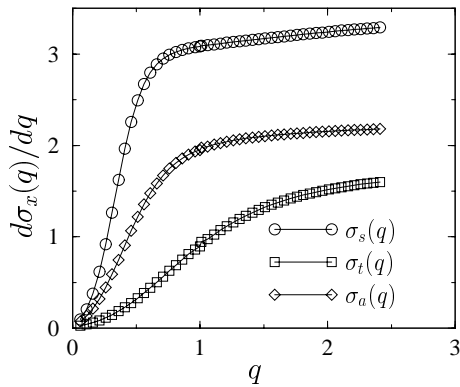
$$\langle x^q \rangle_L = L^{\beta_x(q+1-\tau_x)} \int y^{q-\tau_x} \mathcal{F}(y) dy \sim L^{\beta_x(q+1-\tau_x)}, \quad (2)$$

where we have introduced the transformation  $y = x/L^{\beta_x}$ . In general, one has  $\langle x^q \rangle_L \sim L^{\sigma_x(q)}$ , where the exponent  $\sigma_x(q)$  can be obtained as the slope of the log-log plot of  $\langle x^q \rangle_L$  as a function of  $L$ . If the FSS hypothesis is indeed correct, we expect  $\sigma_x(q) \sim \beta_x(q+1-\tau_x)$ , and therefore one can compute  $\beta_x = d\sigma_x(q)/dq$ . For very small values of  $q$  this is not correct, since the integral in (2) is dominated by its lower cut-off. Once computed the exponent  $\beta_x$ , the corresponding  $\tau_x$  is obtained using the scaling relation  $\sigma_x(1) = \beta_x(2-\tau_x)$ .

In order to apply the moment analysis technique, we have performed simulations of the P-Z and S-Z models in  $d = 2$ , for sizes ranging from  $L = 128$  to  $L = 1024$ . Statistical distributions are obtained by averaging over  $5 \times 10^6$  nonzero avalanches. As a consistency check of our algorithm, we have estimated the average energy of the system in the stationary state, defined by  $\bar{E} = \langle \sum_i E_i \rangle / L^2$ , where the brackets denote an average over at least  $10^6$  stable configurations. Extrapolating to an infinite system size, we obtain a value  $\bar{E} = 0.630 \pm 0.005$  for the P-Z model and  $\bar{E} = 0.626 \pm 0.005$  for the S-Z model, both in good agreement with previous estimates ( $\bar{E} \sim 0.62 - 0.63$ ) [9, 11].

We have computed the moments  $\sigma_x(q)$  for our data from both the P-Z and S-Z models. In the presence of FSS, as we have argued above, one should expect to observe  $d\sigma_x(q)/dq = \beta_x \equiv \text{const}$ . Simulations on the Manna model, reference [7], show that this is indeed the case, with a slope for the moments  $\sigma_x(q)$  that reaches a saturation value for relatively small values of  $q$  [19].

In Figure 1 we have plotted  $d\sigma_x(q)/dq$  for the P-Z model as a function of  $q$ , for  $q < 2.5$ . We observe that all the moments present a noticeable curvature for all  $q$ , and do not seem to reach a constant slope for the values of  $q$  considered. For values of  $q$  larger than 4, the slope of the moment functions seems to finally achieve a saturation value. As an estimate of this trend, in Table 1 we report the value of the local slope at different  $q$ 's, [13], defined by  $\Delta\sigma_x(q) = \sigma_x(q+1) - \sigma_x(q)$ . Assuming that the curvature of the moments at small  $q$  is merely a crossover effect and that the real exponents  $\beta_x$  are given by the saturation



**Fig. 1.** Derivative of the exponents  $\sigma_x(q)$  for the parallel Zhang model. The monotonous increase of the exponent indicates the lack of scaling in this model.

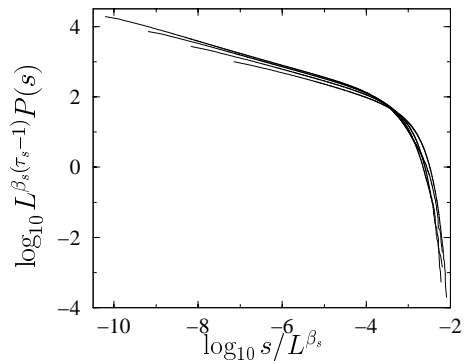
**Table 1.** Local slope of the moment exponents  $\sigma_x(q)$  in the parallel Zhang model, for different values of  $q$ .

$q$	1	2	3	4	5
$\Delta\sigma_s(q)$	3.17	3.30	3.37	3.39	3.40
$\Delta\sigma_t(q)$	1.29	1.61	1.70	1.73	1.74
$\Delta\sigma_a(q)$	2.09	2.19	2.23	2.25	2.26

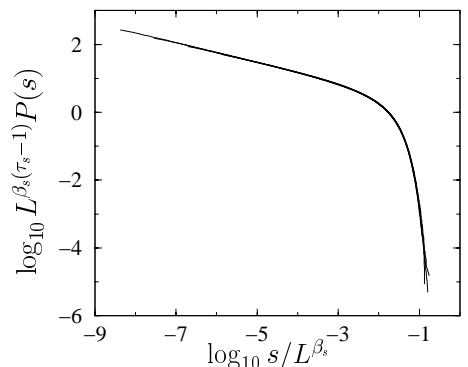
plateaus at large  $q$ , we are led to the values  $\beta_s = 3.39$ ,  $\beta_t = 1.74$ , and  $\beta_a = 2.25$ . In particular, the value of  $\beta_a$  is completely unphysical: the maximum area of an avalanche cannot grow faster than  $L^d$  in  $d$  dimensions, and thus  $\beta_a$  must be smaller than 2. This tells us that the FSS form used in equation (2) is not adequate in the case of the P-Z model, and leads to spurious results. In what respects to the size and time distributions, we can check the validity of this result by means of a data collapse technique: if the FSS ansatz equation (1) is correct, then by rescaling  $x \rightarrow x/L^{\beta_x}$  and  $P(x) \rightarrow L^{\beta_x\tau_x}P(x)$ , we should obtain distributions that collapse onto the same universal curve for different values of  $L$ . In Figure 2 we plot the data collapse of the size distribution  $P(s)$ , with the exponents  $\beta_s = 3.39$  and  $\tau_s = 1.42$ . The really poor collapse of the curves testifies that also the avalanche size distribution does not fulfill the FSS in the P-Z model. We observe a similar lack of collapse for the time distribution.

From Figures 1 and 2, and Table 1, we conclude that the Zhang model with parallel updating violates the FSS hypothesis, equation (1). Noticeably, this lack of FSS is observed also in numerical simulations of the directed version of the Zhang model [20]. We have also tried to fit our data to the more general multifractal scaling form proposed by Kadanoff *et al.* [14], and applied to the BTW model in reference [13]. In this form of multifractal analysis, one tries to collapse the data to the form  $\log(P(x, L))/\log(\alpha L)$  as a function of  $\log(x)/\log(\alpha L)$ , for a suitably chosen constant  $\alpha$ . We have checked that this sort of scaling is also not compatible with our data of the original Zhang model. In particular, we have not succeeded in finding a constant  $\alpha$  for which the scaling is correct.

The moment analysis of the S-Z model yields good results for the size and area distributions, with derivatives of the moments  $\sigma_x(q)$  that reach a saturation plateau for



**Fig. 2.** Data collapse analysis of the integrated distribution of sizes, for the parallel Zhang model. System sizes are  $L = 128, 256, 512, \text{ and } 1024$ . Exponents are  $\beta_s = 3.39$  and  $\tau_s = 1.42$ .



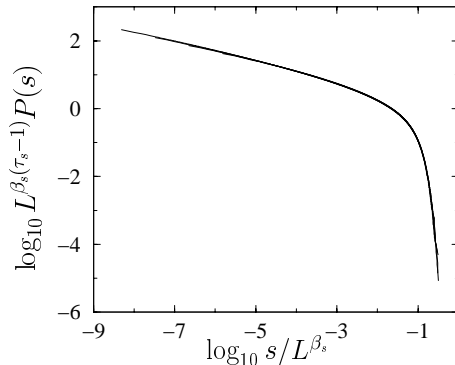
**Fig. 3.** Data collapse analysis of the integrated distribution of sizes for the sequential Zhang model. System sizes are  $L = 128, 256, 512, \text{ and } 1024$ . Exponents are  $\tau_s = 1.29$  and  $\beta_s = 2.78$ .

small values of  $q$ . The values obtained are  $\tau_s = 1.29(2)$  and  $\beta_s = 2.78(2)$  for the size exponents, and in Figure 3 we plot the data collapse analysis for the size distribution. The perfect collapse of this figure confirms the good scaling of this model. The same is obtained with the area distributions with exponents  $\tau_a = 1.43(2)$  and  $\beta_s = 1.94(2)$ . The time distribution, however, shows a lack of scaling, with a not clear trend for the derivative  $\beta_t = d\sigma_t(q)/dq$  as a function of  $q$ . This feature could have several origins included the possibility that in this case we do not have yet reached the scaling regime because of the different time updating, that gives a very small range of time scales.

The complex quality of scaling in the parallel Zhang model can be attributed to the deterministic nature of its dynamical rules, which is somehow smoothed by the stochastic updating in the sequential model. In order to check this conjecture, we propose a variant of the P-Z model, the stochastic parallel Zhang model (SP-Z), in which energy is stochastically redistributed. The model is defined by the following modifications of the relaxation rules. In the SP-Z model, an active site  $k$  loses all its energy,  $E_k \rightarrow 0$ , which is randomly redistributed among its nearest neighbors. In the practical implementation of this rule, we draw four random numbers  $\varepsilon_{k'}$ ,  $0 \leq \varepsilon_{k'} \leq 1$ , with  $\sum_{k'} \varepsilon_{k'} = 1$  and update the nearest neighbors  $k'$  by  $E_{k'} \rightarrow E_{k'} + \varepsilon_{k'} E_k$  [21]. In this model, the stochastic energy input is a random variable uniformly distributed

**Table 2.** Critical exponents for the stochastic parallel Zhang model (SP-Z) and the Manna (M) model. Figures in parenthesis indicate the statistical uncertainty in the last digit. Data for the Manna model from references [6, 7].

Model	$\tau_s$	$\beta_s$	$\tau_t$	$\beta_t$	$\tau_a$	$\beta_a$
SP-Z	1.28(1)	2.76(1)	1.50(2)	1.53(2)	1.35(1)	2.02(2)
M	1.28(1)	2.76(1)	1.48(2)	1.55(1)	1.35(1)	2.02(2)



**Fig. 4.** Data collapse analysis of the integrated distribution of sizes for the stochastic parallel Zhang model. System sizes are  $L = 128, 256, 512,$  and  $1024$ . Exponents from Table 2.

in  $[0, \delta_{\max}]$ , with  $\delta_{\max} = 0.25$ . Sites still have a continuous spectrum of energy, but the new dynamical rules are stochastic. If the assumption that stochasticity yields a standard scaling behavior, then this model should be regarded as the continuous counterpart of the original Manna model [3].

We have performed numerical simulations of the SP-Z model in system sizes ranging from  $L = 128$  to  $L = 1024$ , averaging the probability distributions over  $5 \times 10^6$  nonzero avalanches. We observe that slope of  $\sigma_x(q)$  reaches a saturation value for very small values of  $q$ , for sizes, areas, and also times. In Table 2 we report the values obtained for the exponents  $\beta_x$  and  $\tau_x$ . As expected, the exponents are in perfect agreement with the values found in the Manna model. This fact confirms the presence of a unique universality class for all stochastic models. As a final check, we show in Figure 4 the data collapse analysis for the distribution of sizes. The perfect collapse of these plots should be compared with the poor result obtained in the original Zhang model, shown in Figure 2.

In summary, applying the moment analysis technique, we have shown that the scaling of the Zhang model depends on the updating rules implemented in the simulations. The Zhang model with parallel update violates the FSS hypothesis equation (1) for the avalanche distributions of sizes, times, and areas. The anomalous scaling is stronger than in the BTW model, since data cannot be fitted even to the more general multifractal scaling form. This, contrary to previous claims [5, 10, 11], makes impossible to assign any precise universality class to this model. The sequential updating introduces a small amount of stochasticity in the Zhang model that, in this case, shows FSS for the size and area distributions. In spite of this property, the lack of scaling for the time distribu-

tion does not allow to place the sequential Zhang model in a definite universality class. The anomalous scaling of the original Zhang model can be therefore ascribed to the deterministic nature of the dynamical rules defining the model. A stochastic version of the model shows a standard FSS behavior, compatible with the universality class of the Manna model.

This work has been supported by the European Network under Contract No. ERBFMRXCT980183. We thank A. Stella, A. Vázquez, and S. Zapperi for helpful comments and discussions.

## References

1. H.J. Jensen, *Self-Organized Criticality* (Cambridge University Press, Cambridge, 1998).
2. P. Bak, C. Tang, K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
3. S.S. Manna, J. Phys. A **24**, L363 (1991).
4. D. Dhar, Physica A **263**, 4 (1999).
5. E. Milshtein, O. Biham, S. Solomon, Phys. Rev. E **58**, 303 (1998).
6. A. Chessa, A. Vespignani, S. Zapperi, Comput. Phys. Commun. **121-122**, 299 (1999).
7. S. Lübeck, Phys. Rev. E **61**, 204 (2000).
8. A. Vespignani, R. Dickman, M.A. Muñoz, S. Zapperi, Phys. Rev. E **62**, 4564 (2000).
9. Y.-C. Zhang, Phys. Rev. Lett. **63**, 470 (1989).
10. S. Lübeck, Phys. Rev. E **56**, 1590 (1997).
11. A. Giacometti, A. Díaz-Guilera, Phys. Rev. E **58**, 247 (1998).
12. M. De Menech, A.L. Stella, C. Tebaldi, Phys. Rev. E **58**, R2677 (1998).
13. C. Tebaldi, M. De Menech, A.L. Stella, Phys. Rev. Lett. **83**, 3952 (1999).
14. L.P. Kadanoff, S.R. Nagel, L. Wu, S. Zhou, Phys. Rev. A **39**, 6524 (1989).
15. In the original Zhang model [9] the energy increment  $\delta$  is constant; the random increment used in this work leads to the same critical behavior, but renders the algorithm faster [11].
16. G. Grinstein, in *Scale Invariance, Interfaces and Nonequilibrium Dynamics*, NATO Advanced Study Institute, Series B: Physics, Vol. 344, edited by A. McKane *et al.* (Plenum, New York, 1995); A. Vespignani, S. Zapperi, Phys. Rev. E **57**, 6345 (1998).
17. *Finite Size Scaling*, Vol. 2 of *Current Physics-Sources and Comments*, edited by J.L. Cardy (North Holland, Amsterdam, 1988).
18. In our nomenclature, the exponent  $\beta_s$  is the fractal dimension  $D$  as defined in other publications, while  $\beta_t$  represents the dynamic critical exponent  $z$ .
19. When computing the moments, one has to be aware of the limitations of the validity of equation (2) for large  $q$ . At large  $q$ , the maximum contribution in the average (2) corresponds to the largest avalanches, of which one has the poorest statistics. Therefore, the moment's estimation for large  $q$  is affected by large statistical errors.
20. A. Vázquez, cond-mat/0003420.
21. Note the essential difference between this model and the Manna model in the limit of infinite threshold recently introduced by Lübeck [S. Lübeck, cond-mat/0008304].